GEOS 602 – Geophysical Fields

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Problem Set #3 Due Tue February 16

1. This problem is an exercise in the zonal harmonics. Assume a function $f(\theta)$. We can approximate $f(\theta)$ using a set of k orthogonal functions $P_n(\theta)$),

$$f_k(\theta) = C_0P_0(\theta) + C_1P_1(\theta) + C_2P_2(\theta) + \cdots + C_kP_k(\theta) = \sum_{l=0}^{k} C_l P_l(\theta)$$

We want to choose the coefficients (the C_l) that will minimize the total squared error averaged over the entire spherical surface, given by

$$E = \frac{1}{2}\int_0^{\pi}\left[f(\theta) - f_k(\theta)\right]^2 \sin\theta d\theta$$

Substituting $\mu = \cos\theta$, $d\mu = -\sin\theta\, d\theta$, the integral becomes

$$E = \frac{1}{2}\int_{-1}^{1}\left[f(\mu) - f_k(\mu)\right]^2 d\mu = \frac{1}{2}\int_{-1}^{1}\left[f^2(\mu) - 2f(\mu)f_k(\mu) + f_k^2(\mu)\right]d\mu$$

Substitute the summation above into the expression for E, then use the orthogonality relationship

$$\int_{-1}^{1} P_n(\mu)P_m(\mu) = \begin{cases} A_n, & n = m \\ 0, & n \neq m \end{cases}$$

to simplify the expression. Finally, use the least-squares condition, $\frac{\partial E}{\partial C_l} = 0$, for each C_l to derive

$$C_l = \frac{1}{A_l}\int_{-1}^{1} f(\mu)P_l(\mu)d\mu$$

2. Let $f(\theta,\phi) = A\sin\theta \cdot \cos 2\phi$ on a spherical surface. Find a representation for $f(\theta,\phi)$ in terms of normalized (Schmidt) associated Legendre polynomials. Use Table 6.2 from Blakely (handout).

3. Let $f(\theta,\phi)$ be defined on a sphere such that

$$f(\theta,\varphi) = \begin{cases} 1, & 0 \le \varphi \le \pi \\ 0, & \pi \le \varphi \le 2\pi \end{cases}$$

calculate the degree 0 and 1 (Schmidt-normalized) spherical harmonic coefficients by hand. Using a computer, calculate a ten term approximation to $f(\theta,\phi)$ and plot the approximated values around the equator. You can use either the Schmidt-normalized harmonics or the Kaula-normalized versions for this. See the downloads page for handy routines in MATLAB and FORTRAN.